Math 254B Lecture 29 Notes

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1 Wu's Theorem

1.1 Setup: weaker case and radial Marstrand

Review:

Proposition 1.1. *Fix L with $\dim(K \cap L) > 0$. There exists an ergodic $\tilde{\mu}$ on $K \times (K) \times \mathbb{T}$ such that*

1. *$\widehat{\mu}$ is adapted*
2. *$\tilde{\mu}$ is invariant under $T \times R_{-\xi}$*
3. *$\tilde{\mu}$-a.e. (z, ν, θ) lies in $Z = \{\nu(K \cap L_{z,\theta}) = 1, \dim(\nu) \geq \dim(K \cap L)\}$*

Theorem 1.1 (Furstenberg)**.** *Fix L. For m-ae. $\theta \in \mathbb{T}$, there is a $z \in K$ such that*

$$\dim(K \cap L_{z,\theta}) \geq \dim(K \cap L).$$

Our whole course is set up so we can prove the following result.

Theorem 1.2 (Wu)**.** *For all L, we have*

$$\dim(K \cap L) \leq \max\{0, \dim(K) - 1\}.$$

Let $X = \{(z, \nu) \in K \times \mathbb{P}(K) : z \in \operatorname{supp}(\nu)\}$. Here are the spaces that will play a role in our proofs.

$$\begin{tikzcd}
& (X \times \mathbb{T}, \tilde{\mu}, T \times R_{-\xi}) \arrow[dl, "\varphi"'] \arrow[dr, "\pi"] & \\
(X, \widehat{\mu}, T) \arrow[d, "\psi"] & & (\mathbb{T}, m, T_{-\xi}) \\
(K, \mu, S) & &
\end{tikzcd}$$

First, we will prove the following and reach into the proof to help us prove the big theorem:

Corollary 1.1. *For all L, we have* $\dim(K \cap L) \leq \max\{0, 2\dim(K) - 1\}$.

Proof. Consider $K - K = \{z - w : z, w \in K\}$. Look at slices through this and 0. For all z,

$$(K - K) \cap L_{0,\theta} \supseteq (K - z) \cap L_{0,\theta} = (K \cap L_{z,\theta}) - z.$$

Pick L. Then Furstenburg's theorem gives us that

$$\dim((K - K) \cap L_{0,\theta}) \geq \dim(K \cap L)$$

for m-a.e. θ. To continue, we need a lemma.

Lemma 1.1 (radial Marstrand's slicing theorem)**.** *For any* $A \subseteq \mathbb{R}^2$,

$$\dim(A \cap L_{0,\theta}) \leq \max\{0, \dim(A) - 1\}$$

for m-a.e. θ.

Proof. We can assume A doesn't contain 0. That is, we can assume $A = \bigcup_n (A \cap B(0, 1/n)^c)$. Fix n, then apply a coordinate change. □

Continuing the proof,

Proof. By radial Marstrand, we get

$$\dim(K \cap L) \leq \max(\{0, \dim(K - K) - 1)$$

There is a Lipchitz map from $K \times K$ to $K - K$, so

$$\dim(K \cap L) \leq \max\{0, \dim(K \times K) - 1\}.$$

Since the Hausdorff and box dimension of K agree, we get $\dim(K \times K) = 2\dim(K)$. □

1.2 Proof of Wu's theorem

To prove Wu's theorem, we have to use a better argument.

Proof. Step 1: Assume that $h(\mu, S) = 0$. Then

$$\overline{\dim}(\mu = \frac{h(\mu, S)}{\log(r^{-1})} = 0,$$

so there exists $E \in \mathcal{B}_K$ such that $\mu(E) = 1$ and $\dim(E) = 0$.

Recap: $\tilde{\mu}$ on $K \times P(K) \times \mathbb{T}$ satisfies

(i) $\tilde{\mu}$-a.e. (z, ν, θ) lies in Z.

(ii) μ the first marginal, is supported on E, where $\dim(E) = 0$.

(iii) The last marginal of $\tilde{\mu}$ is m.

Recap: Furstenberg's proof:

- $\tilde{\mu}$-a.e. (z, ν, θ) lies in $Z_W = Z \cap (E \times P(K) \times \mathbb{T})$
- $\implies$ m-a.e. θ lies in $\pi[Z_E]$.
- $\implies$ for m-a.e. $\theta \in \mathbb{T}$, there is a $z \in E$ and $\nu \in P(K)$ such that

$$\nu(K \cap L_{z,\theta}) = 1, \dim(\nu) \geq \dim(K \cap L) \implies \dim(K \cap L_{z,\theta}) \geq \dim(K \cap L).$$

For all $\theta \imath n \mathbb{T}$ and $z \in E$,

$$(K - E) \cap L_{0,\theta} \supseteq (K - z) \cap L_{0,\theta} = (K \cap L_{z,\theta}) - z$$

then

$$\dim((K - E) \cap L_{0,\theta}) \geq \dim(K \cap L)$$

for a.e. θ. So

$$\dim(K \cap L) \leq \max\{0, \dim(K \times E) - 1 = \max\{0, \dim(K) + \cancel{\dim(E)} - 1\}.$$

Step 2: Suppose that $h(\mu, S) > 0$. By Sinai's factor theorem, there exists a factor map $\beta : (K, \mu, S) \to (\Sigma_\ell, p^{\times \mathbb{N}}, \sigma)$ with $h(p^{\times \mathbb{N}}, \sigma) = H(p) = h(\mu, S)$, i.e. $h(\mu, S \mid \beta) = 0$. In total, we have a factor map $(X \times \mathbb{T}, \tilde{\mu}, T \times R_{-\xi}) \to (\Sigma_\ell, p^{\times \mathbb{N}}, \sigma)$. Disintegrate:

$$\tilde{\mu} = \int_{\Sigma_\ell} \tilde{\mu}_\omega \, dp^{\times \mathbb{N}}(\omega).$$

We claim that $\tilde{\mu}_\omega$ satisfies the properties (i),(ii),(ii) for $p^{\times \mathbb{N}}$-a.e. ω.

(i) We knwo that $\tilde{\mu}(Z_0) = 1$ for some $Z_0 \subseteq Z$, where $Z_0 \in \mathcal{B}_{K \times P(K) \times \mathbb{T}}$ and $\tilde{\mu}(Z_0) = \int \tilde{\mu}_\omega(Z_0) \, dp^{\times \mathbb{N}}(\omega)$. This gives $\tilde{\mu}_\omega(Z_0) = 1$ for a.e. ω.

(ii) By uniqueness, if we write

$$\mu = \int_\Sigma \underbrace{\psi_* \varphi_* \tilde{\mu}_\omega}_{\mu_\omega} \, dp^{\times \mathbb{N}}(\omega).$$

this is the disintegration of μ over β. From Lecture 24,

$$\overline{\dim}(\mu_\omega) \leq \frac{h(\mu, S \mid \beta))}{\log(r^{-1})} = 0$$

for a.e. ω.

(iii) Since $(\Sigma_\ell, p^{\times\mathbb{N}}, \sigma)$ is mixing and $(\mathbb{T}, m, R_{-\xi})$ has discrete spectrum, os the only joining is $p^{\times\mathbb{N}}$. So $(\beta \circ \psi \circ \varphi, \pi)_* \tilde{\mu} = p^{\times\mathbb{N}} \times m$. That is, random variables with these distributions must be independent, so

$$\int_{X \times \mathbb{T}} f(\beta(z)) g(\theta) \, d\tilde{\mu}(z, \nu, \theta) = \int_{\Sigma_\ell} dp^{\times\mathbb{N}} \int_{\mathbb{T}} g \, dm$$

for all f and g. We can write the left hand side as

$$\int f(\beta(z)) \cdot \mathbb{E}_{\tilde{\mu}}[g \circ \pi \mid \beta \circ \psi \circ \varphi] \, d\tilde{\mu}.$$

So

$$\int g \circ \pi \, d\tilde{\mu}_\omega = \mathbb{E}_{\tilde{\mu}}[g \circ \pi \mid \beta \circ \psi \circ \varphi](\omega) = \int g \, dm$$

a.e. for all $g \in C(\mathbb{T})$. By separability of $C(\mathbb{T})$, this implies that for a.e. ω, $\int g \circ \pi \, d\widehat{\mu}_\omega = \int g \, dm$ for all $g \in C(\mathbb{T})$. This is precisely property (iii). $\square$

1.3 The real situation

We have been working with a toy version of the situation Wu's result covers. Here is the actual dynamical system we're looking at:

Suppose $T_2[K] = k$ and $T_3[L] = L$. We can't just take the product and run the dynamics, since we've seen before that this changes the "aspect ratio" of boxes, making it hard to calculate Hausdorff dimension. In this case, we have to pick a critical angle θ_0, apply $T_2 \times T_3$ for lines below this angle, and apply $T_2 \times \mathrm{id}$ for lines above this angle. This makes it so the number of times we apply T_3 and T_2 are such that $3^n \sim 2^m$, so the boxes do not get squished so badly. This is the real setting of Wu's result.